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APPROXIMATION OF M/M/1 QUEUE WITH INTERRUPTION SERVICE, RETRIALS AND ORBITAL SEARCH

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Abstract. In this work the M/M/1 queuing system with service interrupted is analysed by the customer himself to take a vacation; during which he joins the orbit. Also, it is assume that the server when it becomes free, can search for customers in orbit. Thus, the access to the server can be done from the queue by primary customers, or from the orbit by secondary customers who have already made at least one pass through the server. In this case: either the customer makes repeated attempts until the server becomes free; or the server goes looking for customers immediately after the completion of each service (assuming that the queue line is free). Given the complexity of the analysis of this model, we use the matrix analytic method, which allows us to obtain an approximation of the limiting probabilities. Some useful performance measures are computed. These results are supported by numerical examples and simulations to study the influence of some parameters on the characteristics of the system.

Keywords: Retrial queue, Interrupted service, Linear retrial policy, Orbital search, Matrix-analytic method.
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1 Introduction

Many considered Retrial queues have a characteristic feature that each service is preceded and followed by an idle period, which is terminated either by the arrival of an orbital customer or a primary customer. Even if there are some customers in the system who want to get service, they cannot occupy the server immediately, because of their ignorance of the server state like the orbital customers, they cannot see the status of the service facility. Therefore, after the completion of each service, next customer enters service only after some time interval during which the server is free, while there may be waiting customers in the orbit, as a selected sample, we mention the works of Arrar et al. (2018, 2012, 2017), Artalejo & Gomez-Corral (2008) and Atencia & Moreno (2006). However, there are some situations in which the server has some initiative searching for blocked customers. Search for customers were discussed by Joshua et al. (2020) and Nila & Sumitha (2022), not only as a mechanism to minimize the idle time of the server or reduce the waiting time for the orbital customers in such models, but also to link and give a combination of the retrial phenomenon and classical queues.

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They were interested in designing retrial queues that reduces the server(s) idle time and achieve this by the introduction of search of orbital customers immediately after a service completion (with search we associate a probability) as follows: after completing a service, the server either immediately picks up a customer from the orbit if any with probability p or remains idle with probability $1 - p$, and in this case, as in the classical retrial queue, a competition takes place in between primary and orbital customers for service. Thus, if the orbital search is done, a service is followed by another service. Otherwise, if no orbital search is done, a service is followed by an idle time. Other related works are done in References: Nila & Sumitha (2020) and Murugan & Vijaykrishnaraj (2019).

Recently, there have been several contributions considering queueing system in which the server may provide a second phase service. Where the customer's service may be viewed as scheduled in two phases: Firstly, all the customers are processed in the first phase. Then, only the customers who qualify are routed in the second phase. The first investigation of retrial queue with second optional service was done by Krishna kumar et al. (2002). They considered an $M/G/1$ retrial queue with second optional service, where at the first phase of service, the server may push out the customer who is receiving such a service, to start the service of another priority arriving customer. The interrupted customers join a retrial queue and the head of this queue is allowed to conduct a repeated attempt in order to start again his essential service after some random time.

Madan (2000) studied such an $M/G/1$ queue with second optional service in which first essential service time follows a general distribution but second optional service is assumed to be exponentially distributed. Medhi (2002) generalized the model by considering that the second optional is also governed by a general distribution. Wang (2004) considers the model with the assumption that the server is subject to breakdowns and repairs in which he assumed that second optional service follows an exponential distribution. As a selection of the related literature, we mention Artalejo & Choudhury (2004), Wang & Zhao (2007), where all customer received batch mode service in a first phase of service followed by individual services in the second phase, and the papers by Ayyappan & Gowthami (2021) and Majid et al. (2021).

However, in this present paper, our purpose is to investigate a much more generalized of a Markovian model by the concept of repeated attempts under a linear retrial policy with orbital search and taking on consideration the interruption service in order to leave the system forever or to rejoin the orbit for another service. We concentrate on the computation of the stationary distribution of the system state, by using of the matrix-analytic methods. This method was developed by Neuts & Rao (1990), Latouche & Ramaswami (1999) and Harchol-Balter (2013), for solving Markov chain that are quite complex. As a related work, the paper by Gómez-Corral (2006) provides a bibliographical guide to the use of the matrix-analytic methods in retrial queueing systems.

The rest of this paper is organized as follows: Section 2 describes the main model in details.The stochastic analysis is performed in section 3 and the special cases for that model are presented. Section 4 is devoted to calculate the stationary distributions by using the Matrix-Analytic Method. Numerical illustration of the model are given in section 5. We finish this work by presenting concluding remarks.

2 Model description

In this model, we analyze an $M/M/1$ retrial queue with customers' break choice and constant retrial policy. We consider a single server retrial queueing system; whose orbit and queue have infinite capacity. We suppose that primary customers arrive according to a Poisson process with rate $\lambda > 0$. The service time is exponentially distributed with parameter μ. The following rules govern the dynamic of the customers:

- If an arriving customer finds the server idle, he immediately begins his service. Otherwise, an arriving customer who finds the server busy, joins the queue line in the service area according to FCFS discipline.

- We assume that a customer who has started his service, may decide to interrupt it and go on vacation or take a break. For this fact, he has to leave the service area and enter the orbit before returning for another service. Thus, the customer can leave the system permanently with probability $(1-p_1)$, after finishing his service, or join the orbit with probability p_1 and return to the server after a period of time.
- We assume that the customers have first access to the orbit after an initial service with rate λp_1, Ayyappan et al. (2010).
- An orbiting customer attempts to access to the server directly at random intervals time (without rejoin the queue line in service area), where the inter-retrials times are exponentialy distributed with rate $\theta > 0$, according to the linear retrial policy $\alpha(1-\delta_{0j})+j\theta$, given that α is a constant rate, δ_{0j} denotes Kronecker function and the rate $j\theta$ is the so-called classical retrial policy rate deppending of the number of customers in the orbit j.
- An orbiting customer can access the server for another service only if the queue is empty.
- The server can go in search of customers immediately after each service completion, by picking up an orbital customer with probability p. The search time is assumed to be negligible. The probability for not going for the search of customers is $q = 1-p$.
- All the random variables defined above are mutually independent.

Adding to the previous parameters, we define the global traffic intensity given by

$$\rho = \frac{\lambda + \lambda p_1}{\mu(1-p_1)}.$$

It is the ratio of the arrival rate $\lambda+\lambda p_1$ to the departure rate $\mu(1-p_1)$, Ayyappan et al. (2010). We can write also, $\rho = \rho_q + \rho_o$, where $\rho_q = \frac{\lambda}{\mu(1-p_1)}$ is the traffic intensity of primary customers and $\rho_o = \frac{\lambda p_1}{\mu(1-p_1)}$ is the traffic intensity of orbiting customers.

3 Stochastic analysis

Denote by $N_q(t)$ the number of customers in the queue line at time t, excluding any customer that might be in service. $N_o(t)$ the number of customers in the orbit at time t. And let $C(t)$ be equal to 0 or 1 depending on the state of the server if it is idle or busy at time t.

Let $N(t)$ denotes the total number of customers in the system at time t (i.e. in orbit, in queue line and in service), where $N(t) = N_q(t) + N_o(t) + C(t)$.

So that the continuous-time stochastic process $\chi = \{C(t), N_q(t), N_o(t); t \geq 0\}$, describes the state of the system with state space $(c,i,j) \in \{0,1\} \times \mathbb{N} \times \mathbb{N}$.

Its infinitesimal transition rates $q_{(0,i,j)(c,m,n)}$ and $q_{(1,i,j)(c,m,n)}$ are given by

- For $i = 0$ and $j = 0$:

$$q_{(0,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,0,0); \\ -\lambda, & \text{if } (c,m,n) = (0,i,j); \\ 0, & \text{otherwise.} \end{cases}$$

and

$$q_{(1,i,j)(c,m,n)} = \begin{cases} \lambda, & \text{if } (c,m,n) = (1,1,0); \\ \lambda p_1, & \text{if } (c,m,n) = (0,0,1); \\ \mu(1-p_1), & \text{if } (c,m,n) = (0,0,0); \\ -[\lambda + \lambda p_1 + \mu(1-p_1)], & \text{if } (c,m,n) = (1,i,j); \\ 0, & \text{otherwise.} \end{cases}$$

- For $i=0$ and $j\geq 1$:

$$q_{(0,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,0,j);\\ \alpha(1-\delta_{0j})+j\theta, & \text{if } (c,m,n)=(1,0,j-1);\\ -[\lambda+\alpha(1-\delta_{0j})+j\theta], & \text{if } (c,m,n)=(0,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

and

$$q_{(1,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,i,j);\\ \lambda p_1, & \text{if } (c,m,n)=(0,0,j+1);\\ \mu(1-p_1)+q\mu, & \text{if } (c,m,n)=(0,0,j);\\ p\mu, & \text{if } (c,m,n)=(1,0,j-1);\\ -[\lambda+\lambda p_1+\mu(1-p_1)+q\mu+p\mu], & \text{if } (c,m,n)=(1,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

- For $j=0$ and $i\geq 1$:

$$q_{(0,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,i-1,0);\\ -\lambda, & \text{if } (c,m,n)=(0,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

and

$$q_{(1,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,i+1,0);\\ \lambda p_1, & \text{if } (c,m,n)=(0,i,1);\\ \mu(1-p_1), & \text{if } (c,m,n)=(0,i,0);\\ -[\lambda+\lambda p_1+\mu(1-p_1)], & \text{if } (c,m,n)=(1,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

- For $i\geq 1$ and $j\geq 1$:

$$q_{(0,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,i-1,j);\\ -\lambda, & \text{if } (c,m,n)=(0,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

and

$$q_{(1,i,j)(c,m,n)}=\begin{cases}\lambda, & \text{if } (c,m,n)=(1,i+1,j);\\ \lambda p_1, & \text{if } (c,m,n)=(0,i,j+1);\\ \mu(1-p_1)+q\mu, & \text{if } (c,m,n)=(0,i,j);\\ p\mu, & \text{if } (c,m,n)=(1,i,j-1);\\ -[\lambda+\lambda p_1+\mu(1-p_1)+q\mu+p\mu], & \text{if } (c,m,n)=(1,i,j);\\ 0, & \text{otherwise.}\end{cases}$$

The stochastic behaviour of the process χ can be represented with the help of the graphical transitions shown in Figure 1.

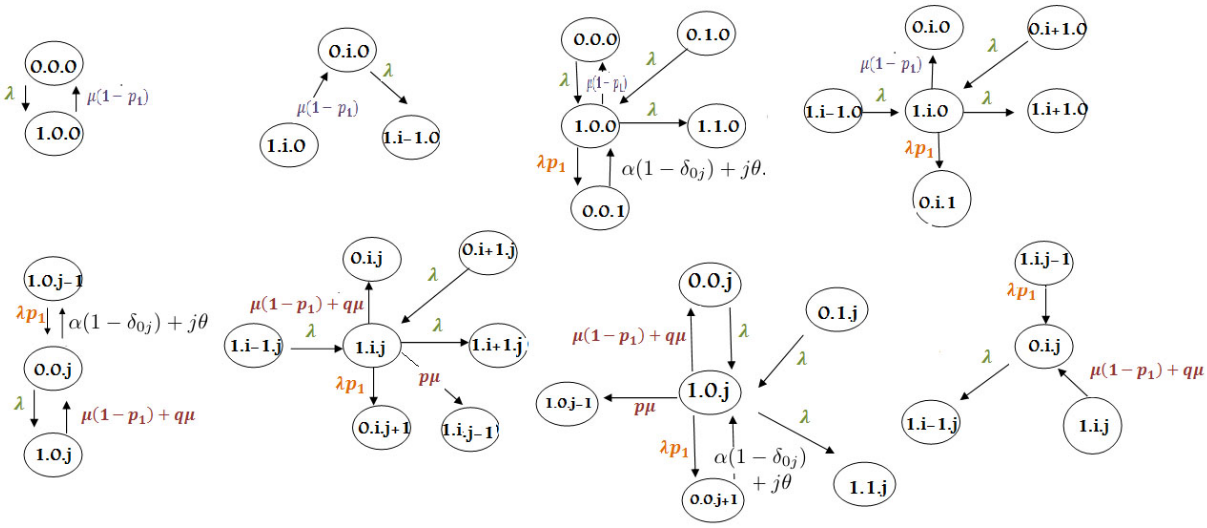

Figure 1: Graphical transitions

Particular Cases

Here, we present some special cases of our model by setting appropriate parameters as follows:

- The main model behaves like an An $M/M/1$ Queue with Retrials After Interruption Service and Orbital Search according to a constant retrial policy if $\theta = 0$;
- The main model behaves like an An $M/M/1$ Queue with Retrials After Interruption Service and Orbital Search according to a classical retrial policy if $\alpha = 0$;
- The main model behaves like an $M/M/1$ standard queue according to First Come, First Served (FCFS) discipline if $p_1 = 0$;
- The main model behaves like an An $M/M/1$ Queue with Retrials After Interruption Service if $p = 0$ (where there is no orbital search). In this case, we can get three cases, depending on the retrial policy that is selected (it can be according to a linear retrial policy or either according to a classical retrial policy when $\alpha = 0$ or constant retrial policy when $\theta = 0$);
- The main model can be without waiting space. Then, if an arriving customer finds the server idle, he immediately begins his service. Otherwise, an arriving customer who finds the server busy, leaves the system without any effect on the system. Its infinitesimal transition rates $q_{(0,n)(c,m)}$ and $q_{(1,n)(c,m)}$ are given by

$$\begin{aligned}
q_{(0,n)(1,n)} &= \lambda, \forall n \geq 0; \\
q_{(0,n)(1,n-1)} &= \alpha(1-\delta_{0n}) + n\theta, \forall n \geq 1; \\
q_{(1,0)(0,0)} &= \mu(1-p_1); \\
q_{(1,n)(0,n)} &= \mu(1-p_1) + q\mu, \forall n \geq 1; \\
q_{(1,n)(1,n-1)} &= p\mu, \forall n \geq 1; \\
q_{(1,n)(0,n+1)} &= \lambda p_1, \forall n \geq 0.
\end{aligned}$$

The set of statistical equilibrium equations for the probabilities $\{\pi_{0,n}, \pi_{1,n}; \forall n \geq 0\}$ have the following expressions

$$\lambda\pi_{0,0} = \mu(1-p_1)\pi_{1,0}; \tag{1}$$
$$[\lambda+\alpha(1-\delta_{0j})+j\theta]\pi_{0,n} = \lambda p_1\pi_{1,n-1}+[\mu(1-p_1)+q\mu]\pi_{1,n}, \ \ \forall n\geq 1; \tag{2}$$
$$\{\lambda p_1+\mu(1-p_1)\}\pi_{1,0} = \lambda\pi_{0,0}+[\alpha(1-\delta_{0j})+j\theta]\pi_{0,1}+p\mu\pi_{1,1}; \tag{3}$$
$$\{\lambda p_1+\mu(1-p_1)+q\mu+p\mu\}\pi_{1,n} = \lambda\pi_{0,n}+[\alpha(1-\delta_{0j})+j\theta]\pi_{0,n+1}+p\mu\pi_{1,n+1}, \ \ \forall n\geq 1; \tag{4}$$

and the normalization equation $\sum_{n\geq 0}\pi_{0,n}+\sum_{n\geq 0}\pi_{1,n}=1$.

In this paper, the retrial models operating under the classical retrial policy or the linear policy have transitions between states $(0,0,j)$ that depend on the third coordinate j. The main analytical difficulties are related to this fact. Since we cannot obtain the steady state distributions of the model in an explicit form. We can solve only one instance of the chain, when the rates are all numbers, by using the Matrix analytic methods, which are approximate numerical methods for solving complex Markov chains.

4 Matrix-Analytic Method

To illustrate the method, it is useful to start by rewriting the balance equations in terms of a "generator matrix", $\mathbf{Q}$. This is a matrix such that

$$\vec{\pi}\ .\ \mathbf{Q}=\vec{0}, \text{ where } \vec{\pi}\ .\ \vec{1}=1. \tag{5}$$

Here, $\vec{\pi}$ is a $2\times(b_1+1)\times(b_2+1)$ row vector of all the limiting distribution probabilities

$$\begin{aligned}\vec{\pi}=&(\pi_{000},\pi_{100},\pi_{001},\pi_{101},...,\pi_{00j},\pi_{10j},\pi_{010},\pi_{110},\pi_{011},\pi_{111},...,\pi_{01j},\pi_{11j},...,\pi_{0ij},\pi_{1ij}),\\ &\forall 0\leq i\leq b_1,\ 0\leq j\leq b_2,\end{aligned} \tag{6}$$

and $\vec{1}$ is an appropriately sized vector of 1s, and $\vec{0}$ denotes a vector with an infinite number of null entries.

Partitioning the limiting probability vector $\vec{\pi}$ as

$$\vec{\pi}=(\vec{\pi}_0,\vec{\pi}_1,...\vec{\pi}_i), \text{ for } 0\leq i\leq b_1, \tag{7}$$

where

$$\vec{\pi}_i=(\pi_{0i0},\pi_{1i0},\pi_{0i1},\pi_{1i1},...,\pi_{0ij},\pi_{1ij}), \text{ for } 0\leq j\leq b_2. \tag{8}$$

By ordering the states as

$$\begin{aligned}S=&\{(0,0,0),(1,0,0),...,(0,0,j),(1,0,j),(0,1,0),(1,1,0),...,(0,1,j),(1,1,j),...\\ &(0,i,0),(1,i,0),...,(0,i,j),(1,i,j)\},\end{aligned}$$

we can express the infinitesimal generator $\mathbf{Q}$ of the process $\{C(t),N_q(t),N_o(t);t\geq 0\}$ in the following matrix block form:

$$\mathbf{Q}=\begin{pmatrix} L_0 & F & & & \\ B & L & F & & \\ & B & L & F & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(1-p_1) & A_0 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S & T & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S & T & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & S & T & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(1-p_1) & A_0 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & V & A_1 & \lambda p_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

With $A_0 = -[\lambda + \lambda p_1 + \mu(1-p_1)]$, $A_1 = -[\lambda + \lambda p_1 + \mu(1-p_1) + q\mu + p\mu]$, $S = \alpha(1-\delta_{0j}) + j\theta$, $T = -[\lambda + \alpha(1-\delta_{0j}) + j\theta]$ and $V = \mu(1-p_1) + q\mu$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

For detailed overviews of the main method, we refer to Artalejo & Gomez-Corral (2008) and Baumann & Sandmann (2010).

5 Numerical examples

We present a numerical example to determine the steady state probabilities $\{(\pi_{0ij}, \pi_{1ij})$, for $0 \le i \le 2$ and $0 \le j \le 3\}$.

Therefore, note that

$$\vec{\pi}_0 = (\pi_{000} \quad \pi_{100} \quad \pi_{001} \quad \pi_{101} \quad \pi_{002} \quad \pi_{102} \quad \pi_{003} \quad \pi_{103});$$

$$\vec{\pi}_1 = (\pi_{010} \quad \pi_{110} \quad \pi_{011} \quad \pi_{111} \quad \pi_{012} \quad \pi_{112} \quad \pi_{013} \quad \pi_{113});$$

$$\vec{\pi}_2 = (\pi_{020} \quad \pi_{120} \quad \pi_{021} \quad \pi_{121} \quad \pi_{022} \quad \pi_{122} \quad \pi_{023} \quad \pi_{123}),$$

also satisfies

$$\vec{\pi}\ .\ \vec{1} = 1;$$
$$\sum_{i=0}^{2} \vec{\pi}_i\ .\ \vec{1} = 1; \qquad \text{(where } \vec{1} \text{ is a } 8 \times 1 \text{ column vector of ones)}$$
$$\sum_{i=0}^{2}\sum_{j=0}^{3} (\pi_{0ij} + \pi_{1ij}) = 1. \tag{9}$$

$$\mathbf{Q} = \begin{pmatrix} L_0 & F & 0 \\ B & L & F \\ 0 & B & L \end{pmatrix}$$

where

$$L_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{V_0} & \mathbf{A_0} & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{S} & \mathbf{T} & \lambda & 0 & 0 & 0 & 0 \\ 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{S} & \mathbf{T} & \lambda & 0 & 0 \\ 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{S} & \mathbf{T} & \lambda \\ 0 & 0 & 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_2} \end{pmatrix},$$

$$L = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{V_0} & \mathbf{A_0} & \lambda p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_1} & \lambda p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & p\mu & \mathbf{V} & \mathbf{A_2} \end{pmatrix},$$

with $\mathbf{S} = \alpha + j\theta$, $\mathbf{T} = -\lambda - \alpha - j\theta$, $\mathbf{V} = \mu(1-p_1) + q\mu$, $\mathbf{V_0} = \mu(1-p_1)$, $\mathbf{A_0} = -[\mu(1-p_1) + \lambda + \lambda p_1]$, $\mathbf{A_1} = -[\mu(1-p_1) + \mu + \lambda + \lambda p_1]$ and $\mathbf{A_2} = -[\mu(1-p_1) + \mu + \lambda]$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Then, based on the matrix analytic method proposed, we briefly provide some numerical examples in some cases that examine the sensitivity and the impact of the system parameters: customers' arrival rate λ, service rate μ, retrial rate θ, α, p and p_1 on the limiting distribution $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \vec{\pi}_2)$. The values of all the parameters were chosen, so that they satisfy the stability condition $\rho < 1$, (Sumitha & Udaya Chandrika, 2012) and the normalizing condition $\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij}) = 1$.

After the above conditions have been verified, we study the behavior of the following performance measures according to the retrial rate θ, the orbital search rate p and the traffic intensity ρ:

- The mean number of customers in the system: $\bar{n} = \sum_{j=0}^{3}\sum_{i=0}^{2}[(i+j)\pi_{0ij} + (i+j+1)\pi_{1ij}]$;
- The mean number of customers in the queue: $\bar{n}_q = \sum_{i=0}^{2} i \sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$;
- The mean number of customers in the orbit: $\bar{n}_o = \sum_{j=0}^{3} j \sum_{i=0}^{2}(\pi_{0ij} + \pi_{1ij})$.

For different values of (p_1, λ) $((0.25, 0.18), (0.50, 1.2), (0.75, 0.04285714))$ and for a fixed value of $\rho = 0.3$, $\mu = 1$, $p = 0.4$ and $\theta = 0.1$, the Table 1 presents the values of $\vec{\pi}$ in case of the linear retrial policy, for $\alpha = 0.05$, Table 3 has the values of $\vec{\pi}$ in case of the classical retrial policy and Table 5 has the values of $\vec{\pi}$ in case of the constant retrial policy for $\alpha = 0.05$.

In a similar way, for another different values of

$$(p_1, \lambda)((0.25, 0.3), (0.50, 0.1666667), (0.75, 0.07142857))$$

and for a fixed value of $\rho = 0.5$, $\mu = 1$, $p = 0.6$ and $\theta = 0.2$, the Tables 2, 4 and 6 present the values of $\vec{\pi}$, respectively, in case of: the linear retrial policy, the classical retrial policy and the constant retrial policy, with $\alpha = 0.1$.

Table 1: Values of $\vec{\pi}$ for the linear retrial policy with $\alpha = 0.05$.

The limiting distribution	$\rho = 0.3$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.4$, $\theta = 0.1$		
	$p_1 = 0.25$ $\lambda = 0.18$	$p_1 = 0.50$, $\lambda = 0.1$	$p_1 = 0.75$, $\lambda = 0.04285714$
π_{000}	0.590148	0.645851	0.7008437
π_{100}	0.1390074	0.1250751	0.1150974
π_{001}	0.03858691	0.0402677	0.02506241
π_{101}	0.004798773	0.00346652	0.001334005
π_{002}	0.0008211582	0.0006755377	0.0001726865
π_{102}	0.000101595	5.737472×10^{-5}	9.051445×10^{-6}
π_{003}	1.285912×10^{-5}	8.154333×10^{-6}	8.443465×10^{-7}
π_{103}	1.661895×10^{-6}	7.279214×10^{-7}	4.796267×10^{-8}
π_{010}	0.1307079	0.1130031	0.1010489
π_{110}	0.03136989	0.02260062	0.01732267
π_{011}	0.01292814	0.01534931	0.01529023
π_{111}	0.0006780884	0.000368091	0.0001158773
π_{012}	0.0002684396	0.0002437705	0.000100796
π_{112}	1.3189×10^{-5}	5.429549×10^{-6}	7.002361×10^{-7}
π_{013}	4.960086×10^{-6}	3.492411×10^{-6}	6.025273×10^{-7}
π_{113}	2.217114×10^{-7}	7.069421×10^{-8}	3.900015×10^{-9}
π_{020}	0.02940598	0.0203566	0.01518462
π_{120}	0.007057436	0.00407132	0.002603077
π_{021}	0.002612059	0.002585267	0.002242
π_{121}	0.0001130267	4.996426×10^{-5}	1.460633×10^{-5}
π_{022}	4.262527×10^{5}	3.193033×10^{-5}	1.256121×10^{-5}
π_{122}	1.915812×10^{-6}	6.316543×10^{-7}	8.09982×10^{-8}
π_{023}	7.153594×10^{-7}	4.032019×10^{-7}	6.969096×10^{-8}
π_{123}	3.152084×10^{-8}	7.943157×10^{-9}	4.508726×10^{-10}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.988683	0.994072	0.996457

Table 2: Values of $\vec{\pi}$ for the linear retrial policy with $\alpha = 0.1$.

The limiting distribution	$\rho = 0.5$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.6$, $\theta = 0.2$.		
	$p_1 = 0.25$ $\lambda = 0.3$	$p_1 = 0.50$, $\lambda = 0.1666667$	$p_1 = 0.75$, $\lambda = 0.07142857$
π_{000}	0.4222393	0.4985866	0.5696219
π_{100}	0.1636481	0.1575284	0.1516412
π_{001}	0.03797358	0.04210704	0.02741862
π_{101}	0.009139597	0.007247313	0.003169886
π_{002}	0.001345373	0.001201436	0.0003476965
π_{102}	0.0003398508	0.0002189052	4.441288×10^{-5}
π_{003}	3.755425×10^{-5}	2.678745×10^{-5}	3.546893×10^{-6}
π_{103}	1.049169×10^{-5}	5.526326×10^{-6}	5.490973×10^{-7}
π_{010}	0.1478462	0.1336726	0.123095
π_{110}	0.0591385	0.04455754	0.03516999
π_{011}	0.0242917	0.03031824	0.03128152
π_{111}	0.002480107	0.001488791	0.0005389042
π_{012}	0.0009658119	0.0009807389	0.0004728451
π_{112}	9.020479×10^{-5}	4.376732×10^{-5}	7.545818×10^{-6}
π_{013}	3.413895×10^{-5}	2.84758×10^{-5}	6.611407×10^{-6}
π_{113}	3.022892×10^{-6}	1.220767×10^{-6}	1.046202×10^{-7}
π_{020}	0.05316112	0.03762091	0.02847453
π_{120}	0.02126445	0.0125403	0.008135581
π_{021}	0.008247441	0.008227445	0.007140646
π_{121}	0.0007646947	0.0003624618	0.0001141715
π_{022}	0.0002930559	0.0002362586	9.996592×10^{-5}
π_{122}	2.657797×10^{-5}	1.019032×10^{-5}	1.575531×10^{-6}
π_{023}	1.016491×10^{-5}	6.681495×10^{-6}	1.386409×10^{-6}
π_{123}	9.183687×10^{-7}	2.93766×10^{-7}	2.250118×10^{-8}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.953352	0.977018	0.9867881

Table 3: Values of $\vec{\pi}$ for the classical retrial policy ($\alpha = 0$).

The limiting distribution	$\rho = 0.3$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.4$, $\theta = 0.1$		
	$p_1 = 0.25$ $\lambda = 0.18$	$p_1 = 0.5$, $\lambda = 1.2$	$p_1 = 0.75$, $\lambda = 0.04285714$
π_{000}	0.5754542	0.6264096	0.6833294
π_{100}	0.138109	0.1252819	0.1171422
π_{001}	0.0518391	0.05630338	0.03674832
π_{101}	0.00614818	0.004542346	0.001746442
π_{002}	0.001259196	0.001072448	0.000276671
π_{102}	0.000149501	0.00008601555	0.0000130069
π_{003}	0.00002151897	0.00001399097	1.39423×10^{-6}
π_{103}	2.667823×10^{-6}	1.17783×10^{-6}	7.052077×10^{-8}
π_{010}	0.1301741	0.1134501	0.1029526
π_{110}	0.03124178	0.02269002	0.01764901
π_{011}	0.01388877	0.01615636	0.01583333
π_{111}	0.000810443	0.000437396	0.0001309193
π_{012}	0.000339369	0.0003001244	0.0001150413
π_{112}	1.823443×10^{-5}	7.402401×10^{-6}	8.496686×10^{-7}
π_{013}	$7.086021 \times \times 10^{-6}$	4.854298×10^{-6}	7.331469×10^{-7}
π_{113}	3.369883×10^{-7}	1.04827×10^{-7}	4.835063×10^{-9}
π_{020}	0.02931635	0.02045363	0.01547427
π_{120}	0.007035924	0.004090726	0.002652733
π_{021}	0.002703665	0.002645083	0.00229322
π_{121}	0.0001259578	0.00005451999	0.0000153111
π_{022}	0.00004982266	0.00003570503	0.00001321699
π_{122}	2.444428×10^{-6}	7.677305×10^{-7}	8.741172×10^{-8}
π_{023}	9.454552×10^{-7}	4.994191×10^{-7}	7.53402×10^{-8}
π_{123}	4.457977×10^{-8}	1.050489×10^{-8}	4.931804×10^{-10}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9886985	0.9940382	0.9963889

Table 4: Values of $\vec{\pi}$ for the classical retrial policy ($\alpha = 0$).

The limiting distribution	$\rho = 0.5$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.6$, $\theta = 0.2$		
	$p_1 = 0.25$ $\lambda = 0.3$	$p_1 = 0.50$, $\lambda = 0.1666667$	$p_1 = 0.75$, $\lambda = 0.07142857$
π_{000}	0.4066091	0.4748591	0.5445161
π_{100}	0.1626436	0.1582863	0.155576
π_{001}	0.05005936	0.05822721	0.0401744
π_{101}	0.01115775	0.009066058	0.003953922
π_{002}	0.00193731	0.001802785	0.0005279602
π_{102}	0.0004515529	0.0002956372	0.00005704345
π_{003}	0.00005594639	0.00004099495	5.210992×10^{-6}
π_{103}	0.00001433503	7.547805×10^{-6}	6.814002×10^{-7}
π_{010}	0.1476159	0.1349189	0.1265567
π_{110}	0.05904637	0.04497297	0.03615906
π_{011}	0.0254885	0.03157541	0.03250287
π_{111}	0.002798325	0.001683134	0.0005916009
π_{012}	0.001120627	0.001126652	0.0005206301
π_{112}	0.0001098381	0.00005279341	8.453775×10^{-6}
π_{013}	0.00004200964	0.00003448311	7.394912×10^{-6}
π_{113}	3.795683×10^{-6}	1.497482×10^{-6}	1.15888×10^{-7}
π_{020}	0.05318374	0.03803282	0.02928964
π_{120}	0.02127349	0.01267761	0.00836847
π_{021}	0.008444061	0.008416814	0.007363395
π_{121}	0.0008153968	0.0003848167	0.0001194553
π_{022}	0.0003188773	0.0002533044	0.0001047189
π_{122}	0.00003000734	0.00001127704	1.662357×10^{-6}
π_{023}	0.00001160231	7.427171×10^{-6}	1.462561×10^{-6}
π_{123}	1.069688×10^{-6}	3.312314×10^{-7}	2.371352×10^{-8}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9532326	0.9767359	0.986407

Table 5: Values of $\vec{\pi}$ for the constant retrial policy ($\theta = 0$).

The limiting distribution	$\rho = 0.3$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.4$, $\alpha = 0.05$		
	$p_1 = 0.25$ $\lambda = 0.18$	$p_1 = 0.50$, $\lambda = 0.1$	$p_1 = 0.75$, $\lambda = 0.04285714$
π_{000}	0.5540986	0.5988841	0.6620737
π_{100}	0.1329836	0.1197768	0.1134984
π_{001}	0.07636424	0.08839677	0.0634247
π_{101}	0.008577415	0.006609705	0.002636794
π_{002}	0.004504865	0.004522928	0.001381255
π_{102}	0.0004815817	0.0003163217	5.118274×10^{-5}
π_{003}	0.0002493836	0.0002137289	2.644291×10^{-5}
π_{103}	2.643485×10^{-5}	1.476659×10^{-5}	9.532396×10^{-7}
π_{010}	0.1259563	0.109028	0.1000165
π_{110}	0.03022951	0.0218056	0.01714569
π_{011}	0.01542753	0.0171366	0.01600702
π_{111}	0.001049354	0.0005667095	0.00015871
π_{012}	0.0006511389	0.0005304503	0.0001613974
π_{112}	5.184005×10^{-5}	2.246323×10^{-5}	2.136046×10^{-6}
π_{013}	3.405675×10^{-5}	2.253655×10^{-5}	2.313148×10^{-6}
π_{113}	2.812898×10^{-6}	1.027721×10^{-6}	3.585448×10^{-8}
π_{020}	0.02842768	0.01969275	0.01504186
π_{120}	0.006822644	0.00393855	0.002578605
π_{021}	0.002821136	0.002651154	0.002250057
π_{121}	0.00014873	6.198894×10^{-5}	1.593798×10^{-5}
π_{022}	8.096822×10^{-5}	5.021091×10^{-5}	1.452213×10^{-5}
π_{122}	5.838095×10^{-6}	1.746949×10^{-6}	1.295113×10^{-7}
π_{023}	3.725683×10^{-6}	1.676963×10^{-6}	1.283736×10^{-7}
π_{123}	3.021545×10^{-7}	7.304443×10^{-8}	1.575133×10^{-9}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9889997	0.9942467	0.9964885

Table 6: Values of $\vec{\pi}$ for the constant retrial policy ($\theta = 0$).

The limiting distribution	$\rho = 0.5$, $\mu = 1$, $\epsilon = 10^{-7}$, $p = 0.4$, $\alpha = 0.1$		
	$p_1 = 0.25$ $\lambda = 0.3$	$p_1 = 0.50$, $\lambda = 0.1666667$	$p_1 = 0.75$, $\lambda = 0.07142857$
π_{000}	0.3906255	0.4520127	0.5256363
π_{100}	0.1562502	0.1506709	0.1501818
π_{001}	0.07058136	0.08794192	0.06748801
π_{101}	0.01435981	0.01210586	0.005421417
π_{002}	0.005877452	0.00653187	0.002294034
π_{102}	0.001107822	0.000814456	0.0001582
π_{003}	0.0004408081	0.00042967	6.565423×10^{-5}
π_{103}	8.107532×10^{-5}	5.189704×10^{-5}	4.276941×10^{-6}
π_{010}	0.1430658	0.1296599	0.1227949
π_{110}	0.05722632	0.04321998	0.03508427
π_{011}	0.02695917	0.03231844	0.03233373
π_{111}	0.003300677	0.001983047	0.0006615969
π_{012}	0.001660531	0.001575287	0.0006234733
π_{112}	0.0002179206	0.0001081043	1.398633×10^{-5}
π_{013}	0.0001132066	8.930036×10^{-5}	1.346917×10^{-5}
π_{113}	1.531995×10^{-5}	6.527447×10^{-6}	3.274094×10^{-7}
π_{020}	0.05174727	0.03668014	0.02845325
π_{120}	0.02069891	0.01222671	0.0081295
π_{021}	0.008584162	0.008326975	0.007196849
π_{121}	0.0008894178	0.0004099293	0.0001208488
π_{022}	0.0004051442	0.0002971419	0.0001082873
π_{122}	4.768429×10^{-5}	1.706986×10^{-5}	1.939634×10^{-6}
π_{023}	2.361115×10^{-5}	1.337758×10^{-5}	1.780166×10^{-6}
π_{123}	3.049585×10^{-6}	8.967875×10^{-7}	3.57627×10^{-8}
$\sum_{i=0}^{2}\sum_{j=0}^{3}(\pi_{0ij} + \pi_{1ij})$	0.9542822	0.9774921	0.986788

On the other hand, it is worthwhile to note that the matrix analytic method proposed in this paper works and is numerically stable one. Moreover, it can be applied on models which satisfy all the previously mentioned conditions.

The impact of the retrial rate θ.

Table 7: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.06227618	1.586191	1.83161
0, 5	0.03062629	1.575898	1.794099
1	0.02344094	1.573528	1.785564
5	0.01666917	1.571285	1.777516
10	0.01574638	1.570979	1.776419
50	0.0149947	1.57073	1.775525
100	0.01489988	1.570698	1.775412

Table 8: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.06415477	1.6579	1.877751
0, 5	0.03302917	1.645738	1.839925
1	0.02636358	1.643106	1.83181
5	0.02021255	1.640671	1.824317
10	0.01938403	1.640342	1.823308
50	0.01871086	1.640075	1.822487
100	0.01862606	1.640042	1.822384

Table 9: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $p = 0.4$.

θ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.04465559	1.705097	1.886251
0, 5	0.02655224	1.694198	1.86227
1	0.02291502	1.692	1.857448
5	0.01963069	1.690014	1.853092
10	0.01919352	1.689749	1.852512
50	0.01883922	1.689534	1.852042
100	0.01879464	1.689507	1.851983

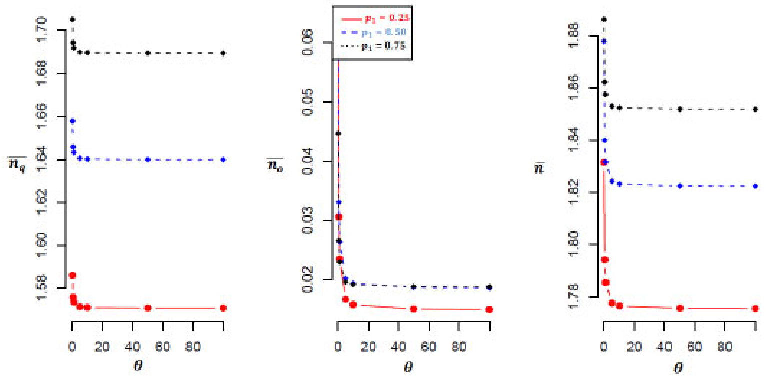

Figure 2: The effect of θ on: $\overline{n}_0$, $\overline{n}_q$ and $\overline{n}$.

The influence of the retrial rate θ is illustrated in Figure 2, from the numerical results listed in Tables 7, 8 and 9. We plot the performance measures by taking $p = 0.4, \rho = 0.3, \mu = 1, \alpha = 0.05$, for the values of $p_1 = 0.25, 0.50$ and 0.75.

We observe that $\overline{n}_q$, $\overline{n}_o$ and $\overline{n}$ decrease when θ increases, with α fixed for several choices of the probability of service interruption and joining the orbit p_1.

The impact of the orbital search rate p.

The effect of the orbital search rate p is shown in Figure 3, from the numerical results listed in Tables 10, 11 and 12, where we have plotted the three performance measures, with respect to p, for $p_1 = 0.25, 0.50$ and 0.75.

Table 10: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.07616402	1.590922	1.848426
0, 2	0.07092634	1.589142	1.84209
0.3	0.06633377	1.587577	1.836528
0.4	0.06227618	1.586191	1.83161
0.5	0.058667	1.584954	1.827232
0.6	0.05543715	1.583844	1.82331
0.7	0.05253092	1.582843	1.819778
0.8	0.04990288	1.581936	1.816582
0.9	0.04751564	1.58111	1.813676

Table 11: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.07341219	1.661547	1.88917
0, 2	0.07006187	1.660228	1.88504
0.3	0.06698661	1.659017	1.881246
0.4	0.06415477	1.6579	1.877751
0.5	0.06153932	1.656868	1.874521
0.6	0.05911702	1.655911	1.871528
0.7	0.05686783	1.655022	1.868748
0.8	0.05477432	1.654193	1.866158
0.9	0.0528213	1.653419	1.863741

Table 12: Performance measures for $\rho = 0.3, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $\theta = 0.1$.

p	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.04835619	1.707146	1.891142
0, 2	0.04706388	1.70643	1.889434
0.3	0.04583162	1.705748	1.887805
0.4	0.04465559	1.705097	1.886251
0.5	0.04353225	1.704477	1.884766
0.6	0.04245836	1.703885	1.883346
0.7	0.04143092	1.703319	1.881988
0.8	0.04044717	1.702777	1.880687
0.9	0.03950454	1.702259	1.879441

We observe that for several choices of the probability of service interruption and joining the orbit p_1, $\overline{n}$, $\overline{n}_o$ and $\overline{n}_q$ always decrease.

The impact of the traffic intensity ρ.

In Figure 4, we choose the values $p_1 = 0.25, 0.50$ and 0.75 to represent the performance measures $\overline{n}_q$, $\overline{n}_o$ and $\overline{n}$, from the numerical results listed in Tables 13, 14 and 15, as functions of the traffic intensity ρ.

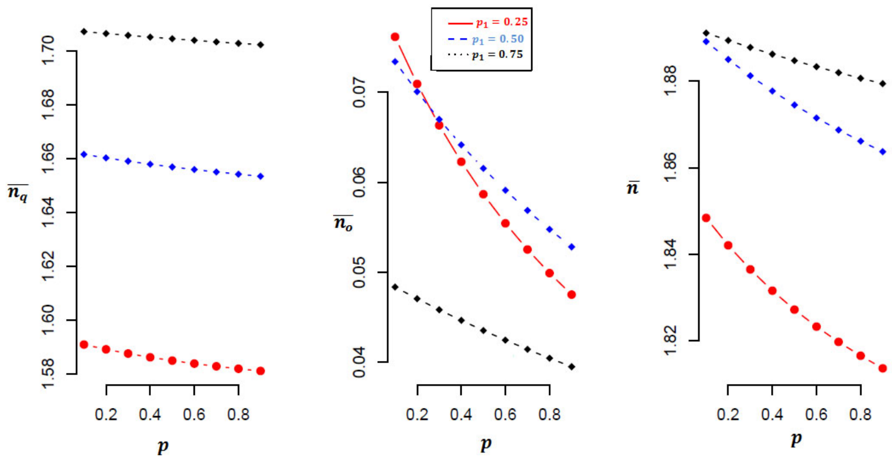

Figure 3: The effect of p on: $\overline{n}_o$, $\overline{n}_q$ and $\overline{n}$.

Table 13: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.25, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.008594285	1.84759	1.929467
0, 2	0.03080721	1.710121	1.874878
0.3	0.06227618	1.586191	1.83161
0.4	0.09979338	1.47379	1.795509
0.5	0.1410306	1.370917	1.763275
0.6	0.184278	1.275856	1.732519
0.7	0.2282345	1.187254	1.701635
0.8	0.271846	1.104098	1.669598
0.9	0.3141844	1.025631	1.635729

Table 14: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.50, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.00873337	1.872918	1.943186
0, 2	0.03157344	1.758953	1.903693
0.3	0.06415477	1.6579	1.877751
0.4	0.1031173	1.568676	1.861936
0.5	0.1460473	1.489864	1.853523
0.6	0.1913093	1.420023	1.850527
0.7	0.2378539	1.357835	1.851594
0.8	0.2850423	1.302153	1.855845
0.9	0.3325048	1.252001	1.862714

We observe that for several choices of p_1, the probability of service interruption and joining the orbit, the mean numbers of customers in the queue $\overline{n}_q$ and in the system $\overline{n}$ respectively, have a decreasing shape with increasing values of ρ, but the main number of customers in orbit $\overline{n_o}$ is strictly an increasing function of ρ.

Table 15: Performance measures for $p = 0.4, \mu = 1, p_1 = 0.75, \alpha = 0.05$ and $\theta = 0.1$.

ρ	$\overline{n}_o$	$\overline{n}_q$	$\overline{n}$
0.1	0.005983884	1.890856	1.949946
0, 2	0.02182542	1.792536	1.912772
0.3	0.04465559	1.705097	1.886251
0.4	0.07212007	1.627936	1.868175
0.5	0.1024134	1.560115	1.856631
0.6	0.1342283	1.500577	1.850082
0.7	0.1666678	1.448272	1.847361
0.8	0.1991511	1.402225	1.847609
0.9	0.2313297	1.361565	1.85021

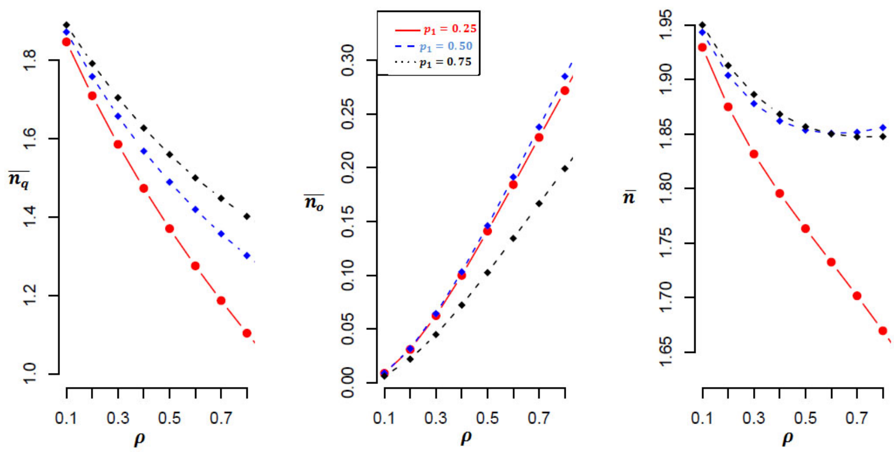

Figure 4: The effect of ρ on: $\overline{n}_o$, $\overline{n}_q$ and $\overline{n}$

6 Conclusion

In this paper a detailed approximation of the stationary distribution is presented for a single-server Markovian queueing model with several parameters, by using the matrix-analytic method. The present investigation includes many features simultaneously such as: (1) retrials according to retrial linear policy; (2) Interruption service; (3) Orbital search. Note that all these realistic assumptions have not been gathered together in the existing literature.

The analytical results have been obtained by using the **Q**-matrix (infinitesimal generating matrix) technique. Particularly, we have obtained approximated values of the steady-state distribution and some performance measures of the model. Moreover, some numerical results are presented to demonstrate how the different parameters of the model influence on the behaviour of the system.

Our study has two main objectives. The first one is to link between the corresponding retrial queue with interruption service under several retrial policy (according to a constant retrial policy, classical retrial policy or linear retrial policy) and the classical queue. That is why our model can be considered as a generalized version of many existing queueing models associated with many practical situations. The second objective is to introduce orbital search in retrial queueing models which allows to minimize the idle time of the server. If the holding costs and cost of using the search of customers will be introduced, the obtained results can be used for the optimal tuning of the parameters of the search mechanism.

This investigation can be further extended by incorporating the batch arrival of primary customers or vacations (breakdowns) of the servers. But the analysis becomes more complicated.

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